Examples of Cardon's Method

Q- Solve $x^3 - 12x + 8\sqrt{2} = 0$

Ans: — The given equation is of 3rd degree equation, so it has three roots.

$$\text{let } x = p^{\frac{1}{3}} + q^{\frac{1}{3}}$$

$$\Rightarrow x^3 - 3p^{\frac{1}{3}}q^{\frac{1}{3}}x - (p+q) = 0 \quad \text{——— (1)}$$

Comparing this with given equation, we get

$$-3p^{\frac{1}{3}}q^{\frac{1}{3}} = -12$$

$$\Rightarrow 4 = p^{\frac{1}{3}}q^{\frac{1}{3}} \Rightarrow pq = 64$$

$$\text{and } -(p+q) = -8\sqrt{2} \Rightarrow p+q = 8\sqrt{2} \quad \text{——— (2)}$$

The quadratic equation from the root p and q be given by

$$z^2 - (p+q)z + pq = 0$$

$$\Rightarrow z^2 - 8\sqrt{2}z + 64 = 0$$

$$z = \frac{8\sqrt{2} \pm \sqrt{(8\sqrt{2})^2 - 4\times1\times64}}{2\times1} = \frac{8\sqrt{2} \pm \sqrt{64\times2 - 64\times4}}{2}$$

$$= \frac{8\sqrt{2} \pm 8\sqrt{2}i}{2} = 4\sqrt{2} \pm 4\sqrt{2}i$$

$$\text{let } p = 4\sqrt{2} + 4\sqrt{2}i \;:\; q = 4\sqrt{2} - 4\sqrt{2}i$$

$$\therefore x = p^{\frac{1}{3}} + q^{\frac{1}{3}} = (4\sqrt{2} + 4\sqrt{2}i)^{\frac{1}{3}} + (4\sqrt{2} - 4\sqrt{2}i)^{\frac{1}{3}} \quad \text{——— (3)}$$

$$\text{let } 4\sqrt{2} = r\cos\theta \text{ and } 4\sqrt{2} = r\sin\theta$$

$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 2\times16\times2$$

$$\Rightarrow r^2 = 64 \;\therefore r = 8 \text{ and } \frac{r\sin\theta}{4\cos\theta} = \frac{4\sqrt{2}}{4\sqrt{2}} = 1$$

$$\Rightarrow \cos\theta = 1 = \tan\frac{\pi}{4} \;\therefore \theta = \frac{\pi}{4}$$

$$\text{By (3), } \quad x = (r\cos\theta + ir\sin\theta)^{\frac{1}{3}} + (r\cos\theta + ir\sin\theta)^{\frac{1}{3}}$$

$$= r^{\frac{1}{3}}\left[\cos\frac{2n\pi+\theta}{3} + i\sin\frac{2n\pi+\theta}{3} + \cos\frac{2n\pi+\theta}{3} - i\sin\frac{2n\pi+\theta}{3}\right]$$ [By De-Moivre's theorem]

$$= r^{\frac{1}{3}} \cdot 2\cos\frac{2n\pi+\theta}{3} \quad \text{Where } n = 0, 1, 2,$$

$$= (8)^{\frac{1}{3}} \cdot 2 \cdot \cos\frac{2n\pi+\pi/4}{3} = 4\cos\frac{8n\pi+\pi}{12}$$

$n = 0, 1, 2,$

$$= 4\cos\frac{\pi}{12}, 4\cos\frac{9\pi}{12}, 4\cos\frac{17\pi}{12}$$

$$= 4\cos\frac{\pi}{12}, 4\cos\frac{3\pi}{4}, 4\cos\frac{17\pi}{12}$$

These are the roots of the given equation.

Q→ Solve $x^3+x^2-16x+20=0$ by Cardon's method.

Ans:— The given equation is of 3rd degree equation. So it has three roots.

Comparing the given cubic equation with standard cubic equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0, \text{ we get}$$

$$a_0=1, \; a_1=\frac{1}{3}, \; a_2=-\frac{16}{3} \text{ and } a_3=20$$

So $H = a_0a_2 - a_1^2 = 1\left(-\frac{16}{3}\right) - \frac{1}{9} = -\frac{49}{9}$

and $G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1$

$$= 1\cdot(20) - 3\cdot 1\cdot \frac{1}{3}\left(-\frac{16}{3}\right) + 2\left(\frac{1}{3}\right)^3$$

$$= 20 + \frac{16}{3} + \frac{2}{27} = \frac{540+144+2}{27} = \frac{686}{27}$$

Now the given cubic equation can be reduced to the standard form

$$t^3 + 3Ht + G = 0$$

$$\Rightarrow t^3 + 3\left(-\frac{49}{9}\right)t + \frac{686}{27} = 0 \Rightarrow t^3 - \frac{49}{3}t + \frac{686}{27} = 0$$

Where $t = a_0x + a_1 = x + \frac{1}{3}$

Let $z = p^{\frac{1}{3}} + q^{\frac{1}{3}}$

Then p and q are the roots of equation,

$$z^2 + 4z - H^3 = 0$$

$$\Rightarrow z^2 + \frac{686}{27}z - \left(-\frac{49}{9}\right)^3 = 0$$

$$\Rightarrow z^2 + \frac{686}{27}z + \frac{49}{729} = 0$$

$$\Rightarrow 729z^2 + 686 \times 27z + 49 = 0 \times 729 = 0$$

$$z = \frac{-686 \times 27 \pm \sqrt{(686 \times 27)^2 - 4 \times 729 \times 49^3}}{2 \times 729}$$

$$= \frac{-686 \pm \sqrt{(686)^2 - 4 \times 49^3}}{2 \times 27}$$

$$= \frac{-686 \pm \sqrt{(686)^2 - (686)^2}}{54} = -\frac{686}{54}$$

$$= -\frac{343}{27} = \left(-\frac{7}{3}\right)^3$$

$\therefore p^{\frac{1}{3}} = -\frac{7}{3}$ and $q^{\frac{1}{3}} = -7/3$

Thus $t_1 = p^{\frac{1}{3}} + q^{\frac{1}{3}} = -\frac{7}{3} - \frac{7}{3} = -\frac{14}{3}$

$t_2 = \omega p^{\frac{1}{3}} + \omega^2 q^{\frac{1}{3}} = (\omega + \omega^2)\left(-\frac{7}{3}\right) = (-1)\left(-\frac{7}{3}\right) = \frac{7}{3}$

$t_3 = \omega^2 p^{\frac{1}{3}} + \omega q^{\frac{1}{3}} = (\omega^2 + \omega)\left(-\frac{7}{3}\right) = (-1)\left(-\frac{7}{3}\right) = \frac{7}{3}$

$\therefore t = x + \frac{1}{3} \Rightarrow x = t - \frac{1}{3}$

$\therefore x = -\frac{14}{3} - \frac{1}{3}, \frac{7}{3} - \frac{1}{3}, \frac{7}{3} - \frac{1}{3} = -5, 2, 2.$

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